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EFFECT OF WARPING RIGIDITY ON STABILITY OF A BAR UNDER ECCENTRIC FOLLOWER FORCE*

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Abstract—Stability of a cantilever under an eccentrically applied follower-end-load is considered. It is shown that a bar with warping rigidity may lose stability by bending-torsional flutter, bending-torsional buckling, transverse flutter, or torsional buckling depending on the eccentricity and other parameters. In particular, it is found that the bending-torsional-flutter-load can be considerably reduced when the bar has warping rigidity. The influence of the internal damping forces on the flutter loads is also assessed.

1. INTRODUCTION

STABILITY of a cantilever subjected at its free end to an eccentric follower force was recently considered by Lin *et al.* [1] who neglected the effects of warping rigidity and internal dissipative forces and found that, for all non-zero values of eccentricity, the bar loses stability by bending-torsional flutter§ only. For zero eccentricity, on the other hand, torsional buckling‖ [2] or transverse flutter [3] may occur. It was conjectured in [1] that warping rigidity may induce additional modes of instability.

In this paper, it is shown that, in general, a cantilever under a follower-end-load may lose stability by torsional buckling, transverse flutter, bending-torsional flutter, or bending-torsional buckling, depending on the parameters of the system; bending-torsional buckling is precluded if no warping rigidity exists. Moreover, the critical-flutter-load may be considerably reduced when the bar possesses some warping rigidity. The influence of the internal dissipative forces on the flutter loads is also assessed [4–6].

2. FORMULATION OF PROBLEM

We consider a cantilevered prismatic thin-walled bar of length L that possesses two axes of symmetry and is subjected at its free end to a compressive follower force P. The force P acts on the axis of least moment of inertia, and remains tangent to the longitudinal fiber at its point of application as the bar deforms (Fig. 1). We assume that the material of the bar obeys a stress–strain relation of the Kelvin-type for both uniaxial and shear deformations, i.e.

$$\sigma = E\varepsilon + E'\dot{\varepsilon}, \qquad \tau = G\gamma + G'\dot{\gamma}, \tag{1}$$

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§ Flutter is defined as oscillations with increasing amplitudes.

‖ Buckling or divergence is defined as attainment of another equilibrium state.

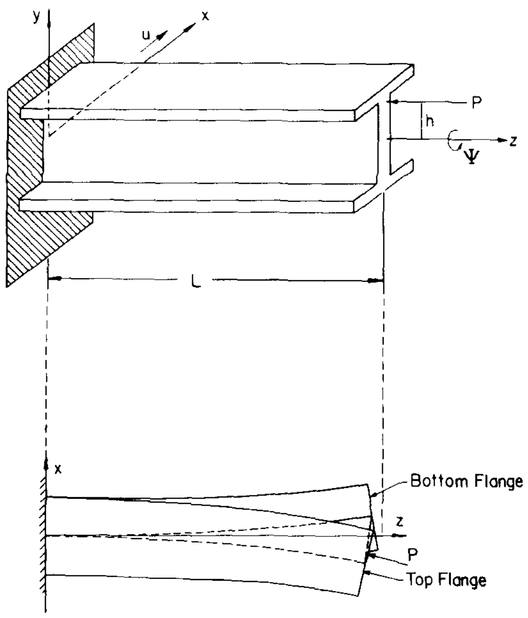

FIG. 1. Cantilever under an eccentric follower force.

where σ is the normal stress, τ is the shear stress, E is the modulus of elasticity, G is the shear modulus, and E' and G' are the coefficients of viscous damping forces. The dots in (1) stand for partial differentiation with respect to time t, and ε and γ are the normal and shear strains, respectively.

Using a system of right-handed rectangular Cartesian coordinates x, y, z, with origin at the fixed end and the z-axis along the centroidal axis of the undeformed bar, we express the strain energy V, the kinetic energy T, the work W of the applied force P, and the energy dissipation D of small bending-torsional vibrations of the bar as follows:

$$V = \tfrac{1}{2}\int_0^L [EIu''^2 + C\Psi'^2 + C_1\Psi''^2]\,\mathrm{d}z,$$

$$T = \tfrac{1}{2}\int_0^L [m\dot{u}^2 + mr^2\dot{\Psi}^2]\,\mathrm{d}z,$$

$$W = \frac{P}{2}\int_0^L [u'^2 + r^2\Psi'^2 - 2hu'\Psi']\,\mathrm{d}z - P\int_{t_0}^{t} [u'(L) - h\Psi'(L)][\dot{u}(L) - h\dot{\Psi}(L)]\,\mathrm{d}t,$$

$$D = \int_{t_0}^{t}\int_0^L [E'I\dot{u}''^2 + C'\dot{\Psi}'^2 + C_1'\dot{\Psi}''^2]\,\mathrm{d}z\,\mathrm{d}t,$$

where $u(z, t)$ and $\Psi(z, t)$ denote, respectively, transverse displacement of the centroidal axis and total rotation of the section at distance z at time t; I is the least moment of inertia, m is mass-density per unit of length, C is the torsional rigidity, and C_1 denotes the warping rigidity of the bar. The force P is assumed to be acting at a distance h from the centroid, and r defines the radius of gyration of the cross-section. The parameters C' and C_1' are given by

$$C' = G'J, \qquad C_1' = E'IC_w,$$

where J is the polar moment of inertia, and C_w is the warping constant of the section [7, 8].

We now consider the total energy H at time t;

$$H = H_0 + V + T + D - W, \tag{2}$$

where H_0 is a constant. We then require that, for the actual motion of the system, the time-rate of change of total energy H vanish identically, yielding

$$\begin{aligned}\frac{dH}{dt} = \int_0^L &\{[EIu^{\cdot\prime\prime}u'' + C\Psi'\Psi^{\cdot\prime} + C_1\Psi^{\cdot\prime\prime}\Psi'']\\ &+ [mu^{\cdot\cdot}u^{\cdot} + mr^2\Psi^{\cdot\cdot}\Psi^{\cdot}] - P[u^{\cdot\prime}u' + r^2\Psi'\Psi^{\cdot\prime} - hu^{\cdot\prime}\Psi' - hu'\Psi^{\cdot\prime}]\\ &+ [E'Iu^{\cdot\prime\prime} + C'\Psi^{\cdot\prime\prime 2} + C_1'\Psi^{\cdot\prime\prime}]\}\, dz + P[u'(L) - h\Psi'(L)][u^{\cdot}(L) - h\Psi^{\cdot}(L)] = 0.\end{aligned}$$

Using integration by parts, we reduce this equation to

$$\begin{aligned}\frac{dH}{dt} = \int_0^L &\{u^{\cdot}[EIu^{\cdot\prime\prime\prime\prime} + mu^{\cdot\cdot} + E'Iu^{\cdot\prime\prime\prime\prime} - Pu'' + Ph\Psi'']\\ &+ \Psi^{\cdot}[C_1'\Psi^{\cdot\prime\prime\prime\prime} + C_1\Psi'''' + Pr^2\Psi'' - C\Psi'' - C'\Psi^{\cdot\prime\prime} - Phu'' + mr^2\Psi'']\}\, dz = 0\end{aligned} \tag{3}$$

with the boundary conditions

$$u = \Psi = u' = \Psi' = 0 \quad \text{at} \quad z = 0$$

$$u'' = \Psi'' = u''' = 0 \quad \text{at} \quad z = L$$

$$\left[P(r^2 - h^2) - C - C'\frac{\partial}{\partial t}\right]\Psi' + \left(C_1 + C_1'\frac{\partial}{\partial t}\right)\Psi''' = 0 \quad \text{at} \quad z = L.$$

We now note that the conservation law stated by (3) is unaltered if arbitrary uniform rigid translation and rigid rotation are imposed on the system. Equation (3) thus yields

$$EIu'''' + E'Iu^{\cdot\prime\prime\prime\prime} + Pu'' - Ph\Psi'' + mu^{\cdot\cdot} = 0$$

$$C_1\Psi'''' + C_1'\Psi^{\cdot\prime\prime\prime\prime} - C'\Psi^{\cdot\prime\prime} + Pr^2\Psi'' - C\Psi'' - Phu'' + mr^2\Psi^{\cdot\cdot} = 0.$$

Introducing the following dimensionless quantities:

$$\zeta = \frac{z}{L}, \qquad \xi = \frac{u}{L}, \qquad \alpha = \frac{h}{r}, \qquad F = \frac{PL^2}{EI}, \qquad \rho = \frac{r}{L},$$

$$\tau = \left[\frac{EI}{mL^4}\right]^{\frac{1}{2}} t, \qquad \delta = \frac{E'}{E}\left(\frac{\tau}{t}\right), \qquad k = \frac{C}{EI},$$

$$\beta = \frac{C_1}{EIL^2}, \qquad \gamma = \frac{C'}{C}\left(\frac{\tau}{t}\right), \qquad \beta\delta = \frac{C_1'}{EIL^2}\left(\frac{\tau}{t}\right),$$

we obtain

$$
\begin{aligned}
\left(1+\delta\frac{\partial}{\partial t}\right)\xi'''' + F\xi'' - F\alpha\rho\Psi'' &= -\ddot{\xi} \\
\beta\left(1+\delta\frac{\partial}{\partial t}\right)\Psi'''' + \left[F\rho^2 - k\left(1+\delta\frac{\partial}{\partial t}\right)\right]\Psi'' - F\rho\alpha\xi'' &= -\rho^2\ddot{\Psi}
\end{aligned}
\tag{4}
$$

with boundary conditions

$$
\begin{aligned}
\xi = \xi' = \Psi = \Psi' &= 0 \quad \text{at} \quad \zeta = 0 \\
\xi'' = \xi''' = \Psi'' &= 0 \quad \text{at} \quad \zeta = 1 \\
\left[F\rho^2(1-\alpha^2) - k\left(1+\gamma\frac{\partial}{\partial t}\right)\right]\Psi' + \beta\left(1+\delta\frac{\partial}{\partial t}\right)\Psi''' &= 0 \quad \text{at} \quad \zeta = 1,
\end{aligned}
\tag{5}
$$

where primes denote partial differentiation with respect to ζ, and dots stand for partial derivatives with respect to τ. We note that if damping forces are neglected, these equations reduce to those obtained in [1] using Hamilton's principle. The present approach, similar to that of [1], demonstrates the effectiveness of the energy method in deducing correct field equations and boundary conditions when nonconservative forces are present (see [9], Section 73, for yet another point of view and other references).

3. STABILITY ANALYSIS

The solution of equation (4) may be taken as

$$
\begin{aligned}
\xi &= e^{i\omega\tau}\sum_{j=1}^{8} A_j e^{\lambda_j\zeta} \\
\Psi &= e^{i\omega\tau}\sum_{j=1}^{8} A_j Q_j e^{\lambda_j\zeta},
\end{aligned}
\tag{6}
$$

where $i = (-1)^{\frac{1}{2}}$,

$$
Q_j = \frac{(1+i\omega\delta)\lambda_j^4 + F\lambda_j^2 - \omega^2}{F\alpha\rho\lambda_j^2},
$$

and $\lambda_j, j = 1, 2, \ldots, 8$, are the roots of the following characteristic equation:

$$
\begin{aligned}
&\beta(1+i\omega\delta)^2\lambda^8 + (1+i\omega\delta)[F(\beta+\rho^2) - k(1+i\omega\gamma)]\lambda^6 \\
&+ [F^2\rho^2(1-\alpha^2) - Fk(1+i\omega\gamma) - \omega^2(1+i\omega\delta)(\rho^2+\beta)]\lambda^4 \\
&- [2F\rho^2\omega^2 - \omega^2 k(1+i\omega\gamma)]\lambda^2 + \rho^2\omega^4 = 0.
\end{aligned}
\tag{7}
$$

The solution (6) must satisfy the boundary conditions (5) which result in a set of eight linear homogeneous equations for A_j. For non-trivial solutions, the determinant of the coefficients of A_j must be zero, yielding

$$
\Delta = |M_{\mu\nu} - P_{\mu\nu}| = 0, \qquad \mu, \nu = 1, 2, 3, 4, \tag{8}
$$

where the elements of Δ are defined as follows: We introduce the notation

$$
\begin{aligned}
K_j &= F\rho^2(1-\alpha^2)-k(1-i\omega\gamma)+\beta(1+i\omega\delta)\lambda_j^2, \\
u_1 &= \lambda_1 = -\lambda_5, \qquad u_2 = \lambda_2 = -\lambda_6, \qquad u_3 = \lambda_3 = -\lambda_7, \\
u_4 &= \lambda_4 = -\lambda_8, \qquad R_2 = Q_2-Q_1, \qquad R_3 = Q_3-Q_1, \\
R_4 &= Q_4-Q_1,
\end{aligned} \tag{9}
$$

and then obtain

$$
\begin{aligned}
M_{11} &= (u_3 Shu_3 - u_1 Shu_1)/R_3 \\
M_{12} &= (u_4 Shu_4 - u_1 Shu_1)/R_4 \\
M_{13} &= (u_3^2 Chu_3 - u_1^2 Chu_1)/R_3 \\
M_{14} &= (u_4^2 Chu_4 - u_1^2 Chu_1)/R_4 \\
M_{21} &= (Q_3 u_3 Shu_3 - Q_1 u_1 Shu_1)/R_3 \\
M_{22} &= (Q_4 u_4 Shu_4 - Q_1 u_1 Shu_1)/R_4 \\
M_{23} &= (Q_3 u_3^2 Chu_3 - Q_1 u_1^2 Chu_1)/R_3 \\
M_{24} &= (Q_4 u_4^2 Chu_4 - Q_1 u_1^2 Chu_1)/R_4 \\
M_{31} &= (u_3^2 Chu_3 - u_1^2 Chu_1)/R_3 \\
M_{32} &= (u_4^2 Chu_4 - u_1^2 Chu_1)/R_4 \\
M_{33} &= (u_3^3 Shu_3 - u_1^3 Shu_1)/R_3 \\
M_{34} &= (u_4^3 Shu_4 - u_1^3 Shu_1)/R_4 \\
M_{41} &= (K_3 Q_3 Chu_3 - K_1 Q_1 Chu_1)/R_3 \\
M_{42} &= (K_4 Q_4 Chu_4 - K_1 Q_1 Chu_1)/R_4 \\
M_{43} &= (K_3 Q_3 u_3 Shu_3 - K_1 Q_1 u_1 Shu_1)/R_3 \\
M_{44} &= (K_4 Q_4 u_4 Shu_4 - K_1 Q_1 u_1 Shu_1)/R_4 \\
P_{11} = P_{12} &= (u_2 Shu_2 - u_1 Shu_1)/R_2 \\
P_{13} = P_{14} &= (u_2^2 Chu_2 - u_1^2 Chu_1)/R_2 \\
P_{21} = P_{22} &= (Q_2 u_2 Shu_2 - Q_1 u_1 Shu_1)/R_2 \\
P_{23} = P_{24} &= (Q_2 u_2^2 Chu_2 - Q_1 u_1^2 Chu_1)/R_2 \\
P_{31} = P_{32} &= (u_2^2 Chu_2 - u_1^2 Chu_1)/R_2 \\
P_{33} = P_{34} &= (u_2^3 Shu_2 - u_1^3 Shu_1)/R_2 \\
P_{41} = P_{42} &= (K_2 Q_2 Chu_2 - K_1 Q_1 Chu_1)/R_2 \\
P_{43} = P_{44} &= (K_2 Q_2 u_2 Shu_2 - K_1 Q_1 u_1 Shu_1)/R_2 .
\end{aligned} \tag{10}
$$

A. *Divergence*

The condition for divergent-type instability is obtained by setting $\omega = 0$ in (8). This yields

$$\Delta = (F-p_1^2)p_2^2 \cos p_1 - (F+p_2^2)p_1^2 Ch p_2, \tag{11}$$

where

$$p_{1,2}^2 = \frac{1}{2\beta}\{\pm[F(\beta+\rho^2)-k]+\sqrt{([F(\beta+\rho^2)-k]^2-4\beta[F^2\rho^2(1-\alpha^2)-Fk])}\}.$$

For given values of β, ρ and k, we now seek the least value of F for which Δ vanishes identically. It is readily seen that this is possible only for $p_2^2 \leq 0$, from which we deduce that

$$F \geq \frac{k}{\rho^2(1-\alpha^2)}. \tag{12}$$

Thus, bending-torsional buckling can occur only for $\alpha^2 < 1$. The solid curve in Fig. 2 is the plot of the critical-buckling-load obtained from (11) for $\beta = 0{\cdot}0005$, $k = 0{\cdot}001$, and $\rho = 0{\cdot}02$.

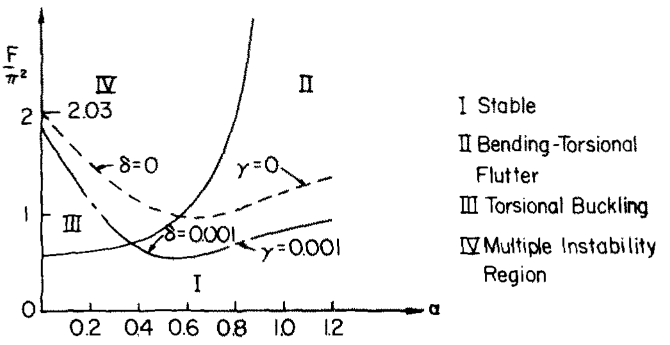

FIG. 2. Critical values of load parameter F/π^2 vs. the eccentricity parameter α for $\rho = 0{\cdot}02$, $k = 0{\cdot}001$, $\beta = 0{\cdot}0005$ and indicated values of damping coefficients δ and γ.

The above results are valid only for non-zero values of α. When $\alpha = 0$, only torsional buckling and transverse flutter are possible. Flutter-type instability occurs for $F \geq 2{\cdot}03\pi^2$ if no dissipation is present. To obtain the torsional-buckling-criterion, we set $\alpha = 0$ in equation (4) and, neglecting the terms which depend on τ, obtain

$$\beta\Psi'''' + (F\rho^2 - k)\Psi = 0 \tag{13}$$

with boundary conditions

$$\begin{aligned} \Psi = \Psi' = 0 \quad &\text{at} \quad \zeta = 0 \\ \Psi'' = (F\rho^2-k)\Psi' + \beta\Psi''' = 0 \quad &\text{at} \quad \zeta = 1. \end{aligned} \tag{14}$$

The solution of (13) now is

$$\Psi(\zeta) = B_1\zeta + B_2 + B_3 \cos\sqrt{(\bar{F})}\zeta + B_4 \sin\sqrt{(\bar{F})}\zeta,$$

where $B_\mu(\mu = 1, 2, 3, 4)$ are constants, and $\bar{F} = (F\rho^2 - k)/\beta$. Using boundary conditions (14), we obtain the characteristic determinant $\Delta_1 = \sqrt{(\bar{F})}\cos\sqrt{(\bar{F})}$ and the following expression for the critical-buckling-load in torsion:

$$F = \frac{\beta\pi^2/4 + k}{\rho^2}. \tag{15}$$

Figure 3 shows various stability regions for $\alpha = 0$ and $\gamma = \delta = 0$.

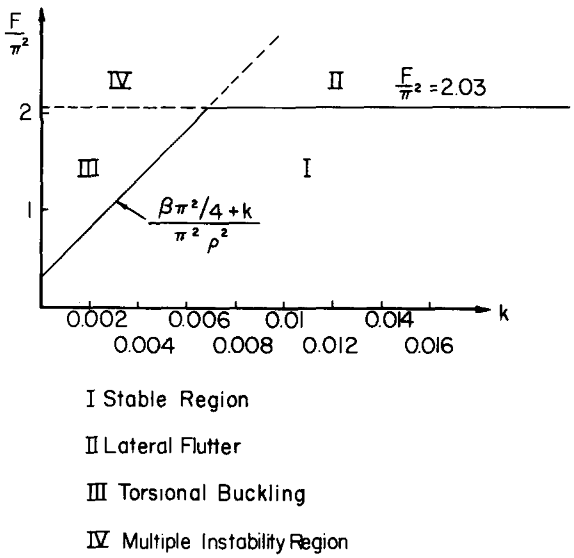

FIG. 3. Stability regions for $\alpha = 0$, $\rho = 0{\cdot}02$, $\beta = 0{\cdot}0005$ and $\gamma = \delta = 0$.

B. *Flutter*

The critical-flutter-load is given by the smallest value of F for which the complex-valued determinant Δ, defined by (8), vanishes for a real value of ω. In actual calculations, the values of k, γ, δ, β, and ρ are first fixed. Then, for various values of F and ω, the roots of the characteristic equation (7) are determined, and the following complex-valued determinant is evaluated:

$$\bar{\Delta} = \Delta/(u_1 - u_2)(u_1 - u_3)(u_1 - u_4)(u_2 - u_3)(u_2 - u_4)(u_3 - u_4), \tag{16}$$

where $u_\mu(\mu = 1, 2, 3, 4)$ are defined by (9). The function $\bar{\Delta}$ rather than Δ is used, since $\bar{\Delta}$ preserves its sign if, for example, u_1 and u_2 are interchanged. The least real values of F and ω for which both the real and imaginary parts of $\bar{\Delta}$ vanish identically now define the flutter load and frequency of the bar, respectively. As is well known, these quantities are highly dependent upon the damping forces in the system. The dashed curve in Fig. 2 is the plot of critical-flutter-load without dissipation, while the one below the dashed curve corresponds to bending-torsional flutter when small internal damping forces are also present.

The bending-torsional-flutter-loads for the bar with and without warping rigidity are compared in Fig. 4. From this figure it is seen that warping rigidity can considerably reduce the critical-flutter-load of the system. It appears that this interesting result has not been noticed before. (The dashed curves in Fig. 4 pertain to bending-torsional buckling instability.)

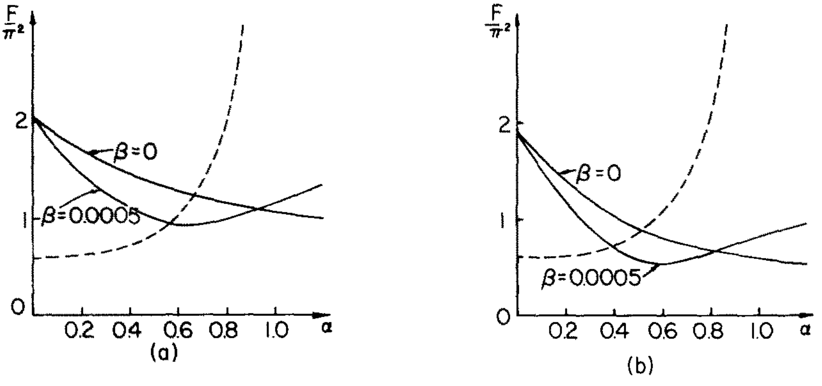

FIG. 4. Critical values of load parameter F/π^2 vs. the eccentricity parameter α for $\rho = 0{\cdot}02$, $k = 0{\cdot}001$ and indicated values of β; (a) $\delta = \gamma = 0$, (b) $\delta = \gamma = 0{\cdot}001$.

We finally note that the values of the parameters used for the illustration, that is $\beta = 0{\cdot}0005$, $k = 0{\cdot}001$, and $\rho = 0{\cdot}02$, may be associated with a wide-flange beam of the following dimensions: web thickness = 0·585, web height = 20, flange thickness = 0·836, flange width = 34, and length = 450, all being measured using an arbitrary unit of length; the Poisson ratio is taken to be $\nu = 0{\cdot}33$.

3. RESULTS AND CONCLUSIONS

From the preceding analysis it is seen that a cantilever with warping rigidity, subjected at its free end to a compressive follower force, may lose stability by either bending-torsional buckling or bending-torsional flutter depending on the eccentricity of the applied load as well as other parameters of the system ($\alpha \neq 0$). For zero eccentricity, $\alpha = 0$, transverse flutter or torsional buckling may occur. Moreover, the critical-flutter-loads are highly dependent upon the damping parameters γ and δ. The present study, therefore, is complementary to the study of Lin *et al.* [1] who neglected the effects of warping rigidity (thus found no bending-torsional buckling) and internal damping forces. An interesting and apparently new result of this study is that warping rigidity may have a destabilizing effect in the bending-torsional flutter mode. Since in the analysis of bending-torsional flutter of airplane wings warping rigidity is neglected, the present finding may have some bearing on the disparity that is often found between the experimental and theoretical flutter studies of wings [10].

REFERENCES

[1] K. H. LIN, S. NEMAT-NASSER and G. HERRMANN, Stability of a bar under eccentric follower force. *J. Engng Mech. Div. Am. Soc. civ. Engrs* **4**, 105–115 (1967).

[2] S. NEMAT-NASSER and G. HERRMANN, Torsional instability of cantilevered bars subjected to nonconservative loading. *J. appl. Mech.* **33**, 102–104 (1966).

[3] M. Beck, Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes. *Z. angew. Math. Phys.* **3**, 225–228 and 476–477 (errata) (1952).
[4] H. Ziegler, Die Stabilitätskriterien der Elastomechanik. *Ing.-Arch.* **20**, 49–56 (1952).
[5] S. Nemat-Nasser and G. Herrmann, Some general considerations concerning the destabilizing effect in nonconservative systems. *Z. angew. Math. Phys.* **17**, 305–313 (1966).
[6] S. Nemat-Nasser, On the stability of the equilibrium of nonconservative continuous systems with slight damping. *J. appl. Mech.* **34**, 344–348 (1967).
[7] F. Bleich, *Buckling Strength of Metal Structures*. McGraw-Hill (1952).
[8] S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, 2nd edition. McGraw-Hill (1961).
[9] V. V. Bolotin, *The Dynamic Stability of Elastic Systems*. Holden Day, (1964) (translated from Russian).
[10] S. N. Prasad, S. Nemat-Nasser and G. Herrmann, Bending-torsional flutter of a swept wing in a high-density low-speed flow. *AIAA Jnl* **5**, 316–321 (1967).

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Абстракт—Исследуется устойчивость консоли под влиянием следящего груза на конце. Указывается, что стержень с искаженной жесткостью терает устойчивость вследствие изгибного крутящего флаттера, изгибного-крутящего выпучивания, поперечного фпаттера илм крутяшего сыпучивансе которые зависят от экцентриситема или других параметров. В качестве особого случая, находится, что нагрузка изгибного-крутяшего флаттера может быть значительно уменьшена, когда стержень обладает искаженной жесткостью. Оценивается также влияние внутренных сил демпфирования на нагрузку флаттера.